To what extent is a large space of matrices not closed under the product?

Clément de Seguins Pazzis *†

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Abstract

Let $\mathbb{K}$ denote a field. Given an arbitrary linear subspace V of $\mathrm{M}_n(\mathbb{K})$ of codimension lesser than $n-1$, a classical result states that V generates the $\mathbb{K}$-algebra $\mathrm{M}_n(\mathbb{K})$. Here, we strengthen this statement in three ways: we show that $\mathrm{M}_n(\mathbb{K})$ is spanned by the products of the form AB with $(A,B) \in V^2$; we prove that every matrix in $\mathrm{M}_n(\mathbb{K})$ can be decomposed into a product of matrices of V; finally, when V is a linear hyperplane of $\mathrm{M}_n(\mathbb{K})$ and $n>2$, we show that every matrix in $\mathrm{M}_n(\mathbb{K})$ is a product of two elements of V.

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1 Introduction

In this paper, $\mathbb{K}$ denotes an arbitrary field, n a positive integer and $\mathrm{M}_n(\mathbb{K})$ the algebra of square matrices of order n with coefficients in $\mathbb{K}$. For $(p,q) \in \mathbb{N}^2$, we also denote by $\mathrm{M}_{p,q}(\mathbb{K})$ the vector space of matrices with p rows, q columns and entries in $\mathbb{K}$. For $(i,j) \in [\![1,n]\!] \times [\![1,p]\!]$, we let $E_{i,j}$ denote the elementary matrix of $\mathrm{M}_{n,p}(\mathbb{K})$ with entry 1 at the (i,j) spot and zero elsewhere. We set $\mathfrak{sl}_n(\mathbb{K}) := \{M \in \mathrm{M}_n(\mathbb{K}) :\ \operatorname{tr} M = 0\}$. The standard lie bracket on $\mathrm{M}_n(\mathbb{K})$ will

*Professor of Mathematics at Lycée Privé Sainte-Geneviève, 2, rue de l'École des Postes, 78029 Versailles Cedex, FRANCE.

†e-mail address: dsp.prof@gmail.com

be written $[-,-]$. We equip $\mathrm{M}_n(\mathbb{K})$ with the non-degenerate symmetric bilinear map $b : (A,B) \mapsto \mathrm{tr}(AB)$. Given a subset $\mathcal{A}$ of $\mathrm{M}_n(\mathbb{K})$, its orthogonal subspace for b will be written $\mathcal{A}^\perp$.

Given a vector space E over $\mathbb{K}$, we let $\mathrm{End}(E)$ denote the ring of linear endomorphisms on E, and, if E is finite-dimensional, we also write $\mathfrak{sl}(E) := \bigl\{u \in \mathrm{End}(E) : \; \mathrm{tr}(u) = 0\bigr\}$.

Here, we will deal with linear subspaces of $\mathrm{M}_n(\mathbb{K})$ with a small *codimension* in $\mathrm{M}_n(\mathbb{K})$ and some properties they share related to the product of matrices. Our starting point is a result that is well-known to specialists of representations of algebras: a strict subalgebra of $\mathrm{M}_n(\mathbb{K})$ must have a codimension greater than or equal to $n-1$. Here is a proof using a theorem of Burnside:

Proof. Let $\mathcal{A}$ be a strict subalgebra of $\mathrm{M}_n(\mathbb{K})$. Choose an algebraic closure $\mathbb{L}$ of $\mathbb{K}$. Then $\mathcal{A}_\mathbb{L} := \mathcal{A} \otimes_\mathbb{K} \mathbb{L}$ is an $\mathbb{L}$-subalgebra of $\mathrm{M}_n(\mathbb{K}) \otimes_\mathbb{K} \mathbb{L}$. Moreover $\dim_\mathbb{L} \mathcal{A}_\mathbb{L} = \dim_\mathbb{K} \mathcal{A}_\mathbb{K}$. Hence $\mathcal{A}_\mathbb{L}$ is a strict subalgebra of $\mathrm{M}_n(\mathbb{K}) \otimes_\mathbb{K} \mathbb{L} \simeq \mathrm{M}_n(\mathbb{L})$, hence Burnside's theorem (see [6] Theorem 1.2.2 p.4) shows that $\mathbb{L}^n$ is not a simple $\mathcal{A}_\mathbb{L}$-module. It follows that we may find a linear embedding of $\mathcal{A}_\mathbb{L}$ into the space of matrices of the form

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \quad \text{with } A \in \mathrm{M}_p(\mathbb{L}),\ B \in \mathrm{M}_{p,n-p}(\mathbb{L}) \text{ and } C \in \mathrm{M}_{n-p}(\mathbb{L}),$$

hence $\operatorname{codim}_{\mathrm{M}_n(\mathbb{L})} \mathcal{A}_\mathbb{L} \geq p(n-p) \geq n-1$. □

As a consequence, if a linear subspace V of $\mathrm{M}_n(\mathbb{K})$ has codimension lesser than $n-1$, then it is not closed under the matrix product, and, better still, V generates the $\mathbb{K}$-algebra $\mathrm{M}_n(\mathbb{K})$. In the present paper, we aim at strengthening this result in various ways.

Notation 1. Given a subset V of $\mathrm{M}_n(\mathbb{K})$, we set

$$V^{(2)} := \bigl\{AB \mid (A,B) \in V^2\bigr\} \quad \text{and} \quad V^{(\infty)} := \bigl\{A_1 A_2 \cdots A_p \mid p \in \mathbb{N},\ (A_1, \dots, A_p) \in V^p\bigr\}$$

i.e. $V^{(\infty)}$ is the *sub-semigroup* of $\bigl(\mathrm{M}_n(\mathbb{K}), \times\bigr)$ generated by V.

Theorem 1. *Let V be a linear subspace of $M_n(\mathbb{K})$ such that* $\operatorname{codim} V < n-1$. *Then every matrix of $M_n(\mathbb{K})$ is a sum of matrices in $V^{(2)}$.*

Notice that

$$W_1 := \left\{ \begin{bmatrix} \alpha & M \\ 0 & L \end{bmatrix} \mid M \in \mathrm{M}_{n-1}(\mathbb{K}),\ L \in \mathrm{M}_{1,n-1}(\mathbb{K}), \alpha \in \mathbb{K} \right\}$$

is a subalgebra of codimension $n-1$ hence the upper bound in Theorem 1 is tight.

Theorem 2. *Let V be a linear subspace of $M_n(\mathbb{K})$ such that* $\operatorname{codim} V < n-1$. *Then V generates the semigroup $\big(M_n(\mathbb{K}), \times\big)$, i.e. $M_n(\mathbb{K}) = V^{(\infty)}$.*

Again, the case of W_1 above shows that the upper bound $n-1$ is tight.

Theorem 3. *Assume $n \geq 3$ and let V be a (linear) hyperplane of $M_n(\mathbb{K})$. Then $M_n(\mathbb{K}) = V^{(2)}$.*

So far, we have not found any linear subspace V of $\mathrm{M}_n(\mathbb{K})$ such that $\operatorname{codim} V < n-1$ and $V^{(2)} \neq \mathrm{M}_n(\mathbb{K})$.

Theorems 1 and 2 will be respectively proven in Sections 2 and 3, whilst Section 4 is devoted to the proof of Theorem 3: there, we will also solve the special case $n = 2$ (i.e. we will determine, up to conjugation, all the hyperplanes H of $\mathrm{M}_2(\mathbb{K})$ for which $H^{(2)} = \mathrm{M}_2(\mathbb{K})$). Those three sections are essentially independent one from the others.

2 The linear subspace spanned by products of pairs

2.1 Products of pairs from the same subspace

Our proof of Theorem 1 is based on the following result:

Proposition 4. *Let V be a linear subspace of $M_n(\mathbb{K})$ such that* $\operatorname{codim} V < n-1$. *Then*

$$\mathfrak{sl}_n(\mathbb{K}) = \operatorname{span}\big\{[A,B] \mid (A,B) \in V^2\big\}.$$

Proof. Set $F := \operatorname{span}\big\{[A,B] \mid (A,B) \in V^2\big\}$. The inclusion $F \subset \mathfrak{sl}_n(\mathbb{K})$ is trivial. Conversely, let $A \in F^\perp$ and $B \in V$. Then, for every $C \in V$, one has $\operatorname{tr}(A[B,C]) = 0$ hence $\operatorname{tr}([A,B]C) = 0$. This shows $\operatorname{ad}_A : M \mapsto [A,M]$ maps V into $V^\perp$. By the rank theorem, we deduce that

$$\dim \operatorname{Ker} \operatorname{ad}_A + \dim V^\perp \geq \dim V$$

hence

$$2\ \mathrm{codim}\, V \geq \mathrm{codim}\,\mathrm{Ker}\,\mathrm{ad}_A\,.$$

Assume that A is not a scalar multiple of the unit matrix I_n. Denote by $P_1, \dots, P_p$ its elementary factors, with $P_p \mid P_{p-1} \mid \cdots \mid P_1$, and $d_i := \deg P_i$. Then the Frobenius theorem on the dimension of the centralizer of a matrix (Theorem 19 p.111 of [2]) shows that

$$\dim \mathrm{Ker}\,\mathrm{ad}_A = \sum_{k=1}^{p} (2k-1)\, d_k = \sum_{1 \leq i,j \leq p} d_{\max(i,j).}$$

Therefore

$$2\,\mathrm{codim}\, V \geq \mathrm{codim}\,\mathrm{Ker}\,\mathrm{ad}_A = \sum_{1 \leq i,j \leq p} \bigl(d_i d_j - d_{\max(i,j)}\bigr) \geq d_1^2 - d_1 + 2 \sum_{i=2}^{p} d_i (d_1 - 1).$$

However $d_1 \geq 2$ since A is not a scalar multiple of I_n, hence

$$2\,\mathrm{codim}\, V \geq \mathrm{codim}\,\mathrm{Ker}\,\mathrm{ad}_A \geq 2d_1 - 2 + 2 \sum_{i=2}^{p} d_i = 2n - 2.$$

This contradicts the initial assumption on V. Hence $F^\perp \subset \mathrm{span}(I_n)$ and therefore $\mathfrak{sl}_n(\mathbb{K}) = \mathrm{span}(I_n)^\perp \subset F$. $\square$

From there, proving Theorem 1 is easy. Let V be a linear subspace of $\mathrm{M}_n(\mathbb{K})$ such that $\mathrm{codim}\, V < n-1$. Then Proposition 4 shows that $\mathfrak{sl}_n(\mathbb{K}) \subset \mathrm{span}\, V^{(2)}$. However, if $\mathfrak{sl}_n(\mathbb{K}) = \mathrm{span}\, V^{(2)}$, then we would have $\forall (A,B) \in V^2,\ \mathrm{tr}(AB) = 0$, hence $V \subset V^\perp$ which would imply that $\mathrm{codim}\, V \geq \frac{n^2}{2}$, in contradiction with the hypothesis $\mathrm{codim}\, V < n-1$. Since $\mathfrak{sl}_n(\mathbb{K})$ is a hyperplane of $\mathrm{M}_n(\mathbb{K})$, this proves $\mathrm{span}\, V^{(2)} = \mathrm{M}_n(\mathbb{K})$.

2.2 Products of pairs from two different subspaces

In this short section, we will diverge slightly from the main theme of this paper. Our aim is the following result, which looks analogous to Theorem 1 but neither generalizes it nor follows from it.

Proposition 5. *Let V and W be two linear subspaces of $M_n(\mathbb{K})$.*

(a) *If* $\operatorname{codim} V + \operatorname{codim} W < n$, *then* $M_n(\mathbb{K})$ *is spanned by* $V \cdot W := \{BC \mid (B,C) \in V \times W\}$.

(b) *If* $\operatorname{codim} V + \operatorname{codim} W = n$ *and* $M_n(\mathbb{K})$ *is not spanned by* $V \cdot W$, *then there is an integer* $p \in [\![0,n]\!]$ *and there are non-singular matrices* P, Q, R *of* $M_n(\mathbb{K})$ *such that*

$$V = P\,V_p\,Q \quad \text{and} \quad W = Q^{-1}\,W_p\,R$$

where, for $k \in [\![0,n]\!]$, *we have set*

$$V_k := \left\{\begin{bmatrix} 0 & L \\ M & N \end{bmatrix} \mid (L,M,N) \in M_{1,n-k}(\mathbb{K}) \times M_{n-1,k}(\mathbb{K}) \times M_{n-1,n-k}(\mathbb{K})\right\}$$

and

$$W_k := \left\{\begin{bmatrix} C & A \\ 0 & B \end{bmatrix} \mid (C,A,B) \in M_{k,1}(\mathbb{K}) \times M_{k,n-1}(\mathbb{K}) \times M_{n-k,n-1}(\mathbb{K})\right\}.$$

Remark 1. A straightforward computation shows that, for every $p \in [\![0,n]\!]$, one has $\operatorname{codim} V_p + \operatorname{codim} W_p = n$ whilst, for every pair $(B,C) \in V_p \times W_p$, the product BC has 0 as entry at the $(1,1)$ spot, hence $E_{1,1}$ is not a linear combination of matrices in $V_p \cdot W_p$.
In particular, this proves that the upper bound in point (a) is tight.

Proof. Assume that $\operatorname{codim} V + \operatorname{codim} W \leq n$. Set $\mathcal{A} := V \cdot W$. We wish to prove that $(V \cdot W)^{\perp} = \{0\}$ save for a few special cases. Let $D \in \mathcal{A}^{\perp}$. Set $B \in V$. Then $\forall C \in W,\ \operatorname{tr}(DBC) = 0$. The linear map

$$f_D : \begin{cases} \mathrm{M}_n(\mathbb{K}) & \longrightarrow \mathrm{M}_n(\mathbb{K}) \\ B & \longmapsto D\,B \end{cases}$$

thus maps V into $W^{\perp}$. However, f_D is represented in a well-chosen basis by the matrix $D \otimes I_n$, with rank $n \operatorname{rk} D$, hence $\dim \operatorname{Ker} f_D = n\,(n - \operatorname{rk} D)$. By the rank theorem, we deduce that

$$\dim V \leq \dim \operatorname{Ker} f_D + \dim W^{\perp} = n\,(n - \operatorname{rk} D) + \operatorname{codim} W$$

hence

$$\operatorname{codim} V + \operatorname{codim} W \geq n \operatorname{rk} D.$$

If $\operatorname{codim} V + \operatorname{codim} W < n$, this shows $D = 0$, hence $\mathcal{A}^\perp = \{0\}$, and we deduce that $\operatorname{span} \mathcal{A} = \mathrm{M}_n(\mathbb{K})$.
Assume now that $\operatorname{codim} V + \operatorname{codim} W = n$ and $\mathcal{A}^\perp \neq \{0\}$, and choose $D \in \mathcal{A}^\perp \smallsetminus \{0\}$. Then $\operatorname{rk} D = 1$. Notice then that $\operatorname{codim} V + \operatorname{codim} W \leq n \operatorname{rk} D$, so the rank theorem shows that $f_D(V) = W^\perp$ and $\operatorname{Ker} f_D \subset V$. A similar line of reasoning shows that

$$g_D : \begin{cases} \mathrm{M}_n(\mathbb{K}) & \longrightarrow \mathrm{M}_n(\mathbb{K}) \\ C & \longmapsto CD \end{cases}$$

satisfies $\operatorname{Ker} g_D \subset W$. Since $\operatorname{rk} D = 1$, there are non-singular matrices P and R such that $D = PE_{1,1}R$. Replacing V and W respectively with RV and WP, we may assume $D = E_{1,1}$. Then the inclusions $\operatorname{Ker} f_D \subset V$ and $\operatorname{Ker} g_D \subset W$ show that V contains every matrix of the form $\begin{bmatrix} 0 \\ M \end{bmatrix}$ for some $M \in \mathrm{M}_{n-1,n}(\mathbb{K})$, and every matrix of the form $\begin{bmatrix} 0 & N \end{bmatrix}$ for some $N \in \mathrm{M}_{n,n-1}(\mathbb{K})$. We may then find linear subspaces E and F respectively of $\mathrm{M}_{1,n}(\mathbb{K})$ and $\mathrm{M}_{n,1}(\mathbb{K})$ such that

$$V = \left\{ \begin{bmatrix} L \\ M \end{bmatrix} \mid L \in E,\ M \in \mathrm{M}_{n-1,n}(\mathbb{K}) \right\} \quad \text{and} \quad W = \left\{ \begin{bmatrix} C & N \end{bmatrix} \mid C \in F,\ N \in \mathrm{M}_{n,n-1}(\mathbb{K}) \right\},$$

with $2n - \dim E - \dim F = \operatorname{codim} V + \operatorname{codim} W$, hence $\dim E + \dim F = n$.
The hypothesis $D \in \mathcal{A}^\perp$ yields $LC = 0$ for every $(L, C) \in E \times F$.
Setting $p := n - \dim E$ and choosing a non-singular matrix Q such that $E\,Q = \left\{ \begin{bmatrix} 0 & L_1 \end{bmatrix} \mid L_1 \in \mathrm{M}_{1,n-p}(\mathbb{K}) \right\}$, we may replace V with $V\,Q$ and W with $Q^{-1}W$.
In this situation, we still have $E_{1,1} \in \mathcal{A}^\perp$, and we now learn that

$$F \subset \left\{ \begin{bmatrix} C_1 \\ 0 \end{bmatrix} \mid C_1 \in \mathrm{M}_{p,1}(\mathbb{K}) \right\}.$$

Since $\dim F = n - p$, we deduce that this inclusion is an equality, which finally shows that $V = V_p$ and $W = W_p$. $\square$

3 The semigroup generated by a large affine subspace

3.1 Starting the induction

We will prove Theorem 2 by establishing the slightly stronger statement:

Theorem 6. *Let $\mathcal{V}$ be an affine subspace of $M_n(\mathbb{K})$ such that* $\operatorname{codim} \mathcal{V} < n - 1$. *Then $M_n(\mathbb{K}) = \mathcal{V}^{(\infty)}$.*

Note that the result trivially holds when $n \leq 2$. We will now proceed by induction. We fix an integer $n \geq 3$ and assume Proposition 6 holds for every affine subspace of $\mathrm{M}_{n-1}(\mathbb{K})$ with a codimension lesser than $n-2$. In the rest of the proof, we fix an affine subspace $\mathcal{V}$ of $\mathrm{M}_n(\mathbb{K})$ such that $\operatorname{codim} \mathcal{V} < n-1$. We let V denote its translation vector space.

3.2 Reduction to the case of non-singular matrices

In this section, we make the following assumption:

Every matrix of $\mathrm{GL}_n(\mathbb{K})$ is a product of matrices of $\mathcal{V}$.

We will prove right away that this entails that every matrix of $\mathrm{M}_n(\mathbb{K})$ is a product of matrices of $\mathcal{V}$. Classically, there are three steps:

(i) $\mathcal{V}$ contains a rank $n-1$ matrix;

(ii) $\mathcal{V}^{(\infty)}$ contains every rank $n-1$ matrix of $\mathrm{M}_n(\mathbb{K})$;

(iii) $\mathcal{V}^{(\infty)}$ contains every singular matrix of $\mathrm{M}_n(\mathbb{K})$.

Proof of step (i). The linear subspace $V^\perp$ has dimension lesser than n hence there is an integer $i \in [\![1,n]\!]$ such that $V^\perp$ contains no non-zero matrix with all columns zero save for the i-th. Conjugating by a permutation matrix, we lose no generality by assuming $V^\perp$ contains no non-zero matrix with all columns zero save for the n-th. This shows that $f : M \mapsto L_n(M)$ is a surjective affine map from $\mathcal{V}$ to $\mathrm{M}_{1,n}(\mathbb{K})$ (where $L_n(M)$ denotes the n-th row of M). Then $\mathcal{W} := f^{-1}\{0\}$ is an affine subspace of $\mathcal{V}$ with $\dim \mathcal{W} = \dim \mathcal{V} - n > n^2 - (2n-1)$. We write then every $M \in \mathcal{W}$ as

$$M = \begin{bmatrix} \alpha(M) \\ 0 \end{bmatrix} \quad \text{with } \alpha(M) \in \mathrm{M}_{n-1,n}(\mathbb{K}).$$

Then $\alpha(\mathcal{W})$ is an affine subspace of $\mathrm{M}_{n-1,n}(\mathbb{K})$ and $\dim \alpha(\mathcal{W}) > n(n-2)$. Using our generalization of Dieudonné's theorem for affine subspaces (cf. Theorem 6 of [7]), we deduce that $\alpha(\mathcal{W})$ contains a rank $n-1$ matrix, hence $\mathcal{V}$ has a rank $n-1$ element. □

Proof of step (ii). Let $A \in \mathrm{M}_n(\mathbb{K})$ be a rank r matrix. If $\mathcal{V}^{(\infty)}$ contains a rank r matrix B, then there are non-singular matrices P and Q such that $A = P\,B\,Q$, hence the preliminary assumption shows that $A \in \mathcal{V}^{(\infty)}$. Step (ii) follows then readily from step (i). □

Proof of step (iii). Let $r \in [\![0, n-1]\!]$. Then the rank r matrix $J_r := \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ decomposes as a product $J_r = \prod_{k=r+1}^{n} (I_n - E_{k,k})$ of rank $n-1$ matrices, hence it belongs to $\mathcal{V}^{(\infty)}$ by step (ii). The argument from step (ii) then shows that $\mathcal{V}^{(\infty)}$ contains every rank r matrix of $\mathrm{M}_n(\mathbb{K})$. □

It now suffices to prove that $\mathrm{GL}_n(\mathbb{K}) \subset \mathcal{V}^{(\infty)}$.

3.3 A good situation

Recall that V denotes the translation vector space of $\mathcal{V}$, and set

$$H := V \cap \operatorname{span}(E_{1,2}, \dots, E_{1,n}).$$

For every $N \in H$, we write

$$N = \begin{bmatrix} 0 & L(N) \\ 0 & 0 \end{bmatrix} \quad \text{with } L(N) \in \mathrm{M}_{1,n-1}(\mathbb{K}).$$

Then $L(H)$ is a linear subspace of $\mathrm{M}_{1,n-1}(\mathbb{K})$ and the rank theorem shows that

$$\dim L(H) = \dim H \geq (n-1) - \operatorname{codim}_{\mathrm{M}_n(\mathbb{K})} V > 0.$$

Hence $L(H)$ contains a non-zero matrix (this will be of crucial interest later on).

Given $M \in \mathrm{M}_n(\mathbb{K})$, we let $C_1(M)$ denote its first column. We consider the affine map

$$(C_1)_{|\mathcal{V}} : \begin{cases} \mathcal{V} & \longrightarrow \mathrm{M}_{n,1}(\mathbb{K}) \\ M & \longmapsto C_1(M). \end{cases}$$

Let us make a first assumption:

(i) $(C_1)_{|\mathcal{V}}$ is onto.

Then

$$\mathcal{W} := \Bigl\{ M \in \mathcal{V} : \quad C_1(M) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T \Bigr\}$$

is an affine subspace of $\mathcal{V}$ with $\dim \mathcal{W} = \dim \mathcal{V} - n$.
For every $M \in \mathcal{W}$, we write

$$M = \begin{bmatrix} 1 & L(M) \\ 0 & K(M) \end{bmatrix} \quad \text{with } K(M) \in \mathrm{M}_{n-1}(\mathbb{K}) \text{ and } L(M) \in \mathrm{M}_{1,n-1}(\mathbb{K}).$$

Finally, we consider the affine subspace $K(\mathcal{W})$ of $\mathrm{M}_{n-1}(\mathbb{K})$. Our second assumption will be:

(ii) $\operatorname{codim}_{\mathrm{M}_{n-1}(\mathbb{K})} K(\mathcal{W}) < n-2$.

From there, we will show that every matrix of $\mathrm{GL}_n(\mathbb{K})$ belongs to $\mathcal{V}^{(\infty)}$. Let $M \in \mathrm{GL}_n(\mathbb{K})$. Then $C_1(M) \neq 0$. We first prove that $C_1(M)$ is also the first column of a non-singular matrix of $\mathcal{V}$:

Lemma 7. *Let $\mathcal{V}'$ be an affine subspace of $M_n(\mathbb{K})$ such that* $\operatorname{codim} \mathcal{V}' < n-1$. *Let $C \in M_{n,1}(\mathbb{K}) \smallsetminus \{0\}$ and assume some element of $\mathcal{V}'$ has C as first column. Then some element of $\mathcal{V}' \cap GL_n(\mathbb{K})$ has C as first column.*

Proof. Set $C_0 := \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T$. Choosing $P \in \mathrm{GL}_n(\mathbb{K})$ such that $PC = C_0$ and replacing $\mathcal{V}'$ with $P\mathcal{V}'$, we may assume $C = C_0$. With the above notations (though not assuming that $N \mapsto C_1(N)$ maps $\mathcal{V}'$ onto $\mathrm{M}_{n,1}(\mathbb{K})$), we obtain that $\mathcal{W}' \neq \emptyset$, hence the rank theorem shows $\operatorname{codim}_{\mathrm{M}_{n-1}(\mathbb{K})} K(\mathcal{W}') < n-1$. Dieudonné's theorem for affine subspaces [1] then shows that the affine subspace $K(\mathcal{W}')$ contains a non-singular matrix, QED. $\square$

From there, we may choose some $N \in \mathcal{V} \cap \mathrm{GL}_n(\mathbb{K})$ with $C_1(M)$ as first column. The matrix $A := N^{-1}M$ is then non-singular and has the form

$$A = \begin{bmatrix} 1 & * \\ 0 & P \end{bmatrix} \quad \text{for some } P \in \mathrm{GL}_{n-1}(\mathbb{K}).$$

It thus suffices to prove that $A \in \mathcal{V}^{(\infty)}$. This will come from the next proposition:

Proposition 8. *Assuming conditions (i) and (ii) hold, let $P \in GL_{n-1}(\mathbb{K})$ and $L \in M_{1,n-1}(\mathbb{K})$. Then the matrix $\begin{bmatrix} 1 & L \\ 0 & P \end{bmatrix}$ belongs to $\mathcal{W}^{(\infty)}$.*

Proof. Condition (ii) and the induction hypothesis yield matrices $P_1, \dots, P_r$ in $K(\mathcal{W})$ such that $P = P_1 P_2 \cdots P_r$, hence there are row matrices $L_1, \dots, L_r$ in $\mathrm{M}_{1,n-1}(\mathbb{K})$ such that:

- $Q_k := \begin{bmatrix} 1 & L_k \\ 0 & P_k \end{bmatrix}$ belongs to $\mathcal{V}$ for every $k \in [\![1, r]\!]$;
- $Q_1 Q_2 \cdots Q_r = \begin{bmatrix} 1 & L' \\ 0 & P \end{bmatrix}$ for some $L' \in \mathrm{M}_{1,n-1}(\mathbb{K})$.

In order to conclude, it suffices to prove that the matrix $\begin{bmatrix}1 & L-L' \\ 0 & I_{n-1}\end{bmatrix}$ belongs to $\mathcal{V}^{(\infty)}$, since left-multiplying it by $\begin{bmatrix}1 & L' \\ 0 & P\end{bmatrix}$ yields $\begin{bmatrix}1 & L \\ 0 & P\end{bmatrix}$.

We actually prove that $\mathcal{V}^{(\infty)}$ contains $\begin{bmatrix}1 & L_1 \\ 0 & I_{n-1}\end{bmatrix}$ for every $L_1 \in \mathrm{M}_{1,n-1}(\mathbb{K})$. Notice that the set $\mathcal{A}$ of those $L_1 \in \mathrm{M}_{1,n-1}(\mathbb{K})$ such that $\begin{bmatrix}1 & L_1 \\ 0 & I_{n-1}\end{bmatrix} \in \mathcal{V}^{(\infty)}$ is closed under sum because $\mathcal{V}^{(\infty)}$ is closed under product.

Let $R \in \mathrm{GL}_n(\mathbb{K})$. By the previous line of reasoning, there are matrices $Q_1 = \begin{bmatrix}1 & L_1 \\ 0 & P_1\end{bmatrix}, \dots, Q_r = \begin{bmatrix}1 & L_r \\ 0 & P_r\end{bmatrix}$ in $\mathcal{W}$ and a row matrix $L' \in \mathrm{M}_{1,n-1}(\mathbb{K})$ such that $Q_1 \cdots Q_r = \begin{bmatrix}1 & L' \\ 0 & R^{-1}\end{bmatrix}$. Also, there is a row matrix $L'' \in \mathrm{M}_{1,n-1}(\mathbb{K})$ such that $\begin{bmatrix}1 & L'' \\ 0 & R\end{bmatrix}$ belongs to $\mathcal{W}^{(\infty)}$.

Notice that L_r may be replaced with $L_r + L_0$ for any $L_0 \in L(H)$ (recall the definition of $L(H)$ from the beginning of the section): it follows that $\begin{bmatrix}1 & L'+L_0 \\ 0 & R^{-1}\end{bmatrix} \in \mathcal{V}^{(\infty)}$ for any $L_0 \in L(H)$. Right-multiplying this matrix by $\begin{bmatrix}1 & L'' \\ 0 & R\end{bmatrix}$, we deduce that $L'R + L'' + L_0R$ belongs to $\mathcal{A}$ for every $L_0 \in L(H)$. We have thus found, for every $R \in \mathrm{GL}_n(\mathbb{K})$, a row matrix $L_R \in \mathrm{M}_{1,n-1}(\mathbb{K})$ such that $L_R + L(H)\,R \subset \mathcal{A}$. Recall from the beginning of this paragraph that there is a non-zero $E \in L(H)$. We may then find non-singular matrices $P_1, \dots, P_{n-1}$ such that $(EP_i)_{1 \leq i \leq n-1}$ is a basis of $\mathrm{M}_{1,n-1}(\mathbb{K})$. Since $\mathcal{A}$ is closed under addition and $L(H)$ is a linear subspace of $\mathrm{M}_{1,n-1}(\mathbb{K})$, we deduce that $\mathcal{A}$ contains $\sum_{k=1}^{n-1} L_{P_k} + \operatorname{span}(EP_k)_{1 \leq k \leq n-1}$, which clearly equals $\mathrm{M}_{1,n-1}(\mathbb{K})$. Hence $\mathcal{A} = \mathrm{M}_{1,n-1}(\mathbb{K})$, QED. $\square$

3.4 Why the good situation almost always arises up to conjugation

Notice first that given $P \in \mathrm{GL}_n(\mathbb{K})$, one has $(P\mathcal{V}P^{-1})^{(\infty)} = P\,\mathcal{V}^{(\infty)}\,P^{-1}$, so we may replace $\mathcal{V}$ with any conjugate affine subspace in order to prove that $\mathcal{V}^{(\infty)} = \mathrm{M}_n(\mathbb{K})$. We denote by $(e_1, \dots, e_n)$ the canonical basis of $\mathbb{K}^n$.

Here, we prove the following result:

Proposition 9. *Let $\mathcal{V}$ be an affine subspace of $M_n(\mathbb{K})$ such that* $\operatorname{codim} \mathcal{V} < n-1$. *Then :*

(a) Either $n = 3$ and there exists $a \in \mathbb{K}$ such that $\mathcal{V} = \{M \in M_3(\mathbb{K}) : \operatorname{tr} M = a\}$;

(b) Or there exists $P \in GL_n(\mathbb{K})$ such that $P\,\mathcal{V}\,P^{-1}$ satisfies conditions (i) and (ii) of Section 3.3.

Before proving this, we must analyze condition (i) in terms of the structure of $V^\perp$, where V denotes the translation vector space of $\mathcal{V}$. For $M \mapsto C_1(M)$ not to be onto from $\mathcal{V}$, it is necessary and sufficient for it not to be onto from V, which is equivalent to the existence of a non-zero row matrix $L \in \mathrm{M}_{1,n}(\mathbb{K})$ such that $\begin{bmatrix} L \\ 0 \end{bmatrix} \in V^\perp$. Hence (i) holds if and only if no matrix A in $V^\perp$ satisfies $\operatorname{Im} A = \operatorname{span}(e_1)$.
Assume now that condition (i) holds. The rank theorem shows:

$$\operatorname{codim}_{\mathrm{M}_{n-1}(\mathbb{K})} K(\mathcal{W}) \leq \operatorname{codim}_{\mathrm{M}_n(\mathbb{K})} \mathcal{V} < n-1.$$

If (ii) does not hold, then the rank theorem shows that $\operatorname{codim}_{\mathrm{M}_n(\mathbb{K})} \mathcal{V} = n-2$ and $\dim L(H) = n-1$, hence $L(H) = \mathrm{M}_{1,n-1}(\mathbb{K})$: it would follow that V contains every matrix $A \in \mathfrak{sl}_n(\mathbb{K})$ such that $\operatorname{Im} A = \operatorname{span}(e_1)$.
We deduce that conditions (i) and (ii) hold in the case $V^\perp$ contains no rank 1 matrix with image $\operatorname{span}(e_1)$ *and* V does not contain every matrix $A \in \mathfrak{sl}_n(\mathbb{K})$ with image $\operatorname{span}(e_1)$. With that in mind, we may now prove Proposition 9.

Proof of Proposition 9. We reason in terms of linear operators. We use the canonical basis to identify $\mathcal{V}$ with an affine space of linear endomorphisms of $\mathbb{K}^n$. The symmetric bilinear form $(A, B) \mapsto \operatorname{tr}(AB)$ on $\mathrm{M}_n(\mathbb{K})$ then corresponds to $(u, v) \mapsto \operatorname{tr}(u \circ v)$.
We assume there is no $P \in \mathrm{GL}_n(\mathbb{K})$ such that $P\,\mathcal{V}\,P^{-1}$ satisfies conditions (i) and (ii) of Section 3.3. By the above remarks, this shows that for every 1-dimensional linear subspace $D \subset \mathbb{K}^n$ for which $V^\perp$ contains no endomorphism with image D, one has $u \in V$ for every $u \in \mathfrak{sl}(\mathbb{K}^n)$ such that $\operatorname{Im} u = D$.
We then wish to show that V contains every trace 0 endomorphism.

- Consider the linear subspace U of $V^\perp$ spanned by its rank 1 endomorphisms. In U, we choose a basis $(u_1, \dots, u_r)$ consisting of rank 1 endomorphisms, and we set $F := \operatorname{Im} u_1 + \cdots + \operatorname{Im} u_r \subset \mathbb{K}^n$. Then every rank 1

element in $V^\perp$ has its image included in F and

$$\dim F \leq r \leq \dim V^\perp \leq n-2.$$

- It follows that V contains every $u \in \mathfrak{sl}(\mathbb{K}^n)$ such that $\operatorname{rk} u = 1$ and $\operatorname{Im} u \not\subset F$. We will let $\mathcal{B}$ denote the set of those endomorphisms.

- Notice that the set of rank 1 endomorphisms of $\mathbb{K}^n$ with trace 0 spans $\bigl\{u \in \operatorname{End}(\mathbb{K}^n) : \ \operatorname{tr} u = 0\bigr\}$: it suffices to consider the matrices $E_{i,j}$ and $E_{j,i}$, for $1 \leq i < j \leq n$, and the matrices $E_{1,1} + E_{k,1} - E_{1,k} - E_{k,k}$, for $2 \leq k \leq n$.

- We finish by proving that every $u \in \operatorname{End}(\mathbb{K}^n)$ with rank 1 and trace 0 is a linear combination of elements of $\mathcal{B}$. Set $u \in \operatorname{End}(\mathbb{K}^n)$ such that $\operatorname{rk} u = 1$, $\operatorname{tr} u = 0$ and $\operatorname{Im} u \subset F$. Choose $x_1 \in \operatorname{Im} u \smallsetminus \{0\}$. Since $\operatorname{codim} F \geq 2$, we may choose $x_2 \in E \smallsetminus (F \cup \operatorname{Ker} u)$ and then $x_3 \in E$ such that $\operatorname{span}(x_2, x_3) \cap F = \{0\}$. We finally extend (x_1, x_2, x_3) into a basis $\mathbf{B}$ of $\mathbb{K}^n$ using vectors of $\operatorname{Ker} u$.

  Then there is a matrix $A \in \mathrm{M}_3(\mathbb{K})$, of the form $A = \begin{bmatrix} 0 & L \\ 0 & 0 \end{bmatrix}$ for some $L \in \mathrm{M}_{1,2}(\mathbb{K}) \smallsetminus \{0\}$, such that

  $$M_{\mathbf{B}}(u) = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}.$$

  Since $\operatorname{span}(x_1, x_2, x_3) \cap F = \operatorname{span}(x_1)$, we deduce: for every $A_1 \in \mathfrak{sl}_3(\mathbb{K})$ such that $\operatorname{rk} A_1 = 1$ and $\operatorname{Im} A_1 \neq \operatorname{span} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, there is some $v \in \mathcal{B}$ such that $M_{\mathbf{B}}(v) = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$. In order to conclude, it thus suffices to solve the case $n = 3$.
  By a change of basis, it suffices to prove that the vector space $\mathfrak{sl}_3(\mathbb{K})$ is spanned by its rank 1 matrices whose image is different from $\operatorname{span} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$. This is obvious using the family from the preceding bullet-point.

Finally, we have shown that $\mathfrak{sl}_n(\mathbb{K}) \subset V$. If $V = \mathrm{M}_n(\mathbb{K})$, then conditions (i) and (ii) of Section 3.3 obviously hold. If not, one has $\mathfrak{sl}_n(\mathbb{K}) = V$ thus $\mathcal{V} = \bigl\{M \in \mathrm{M}_n(\mathbb{K}) : \ \operatorname{tr} M = a\bigr\}$ for some $a \in \mathbb{K}$. Then condition (i) is clearly satisfied by $\mathcal{V}$, and since (ii) is not, one has $\operatorname{codim}_{\mathrm{M}_n(\mathbb{K})} \mathcal{V} = n - 2$ (see the remarks above the present proof). Since $\mathcal{V}$ is a hyperplane of $\mathrm{M}_n(\mathbb{K})$, we finally deduce that $n = 3$. $\square$

3.5 The exceptional case

Combining Proposition 9 with the arguments from Sections 3.2 and 3.3, it is clear that our proof of Theorem 6 will be complete when the following result will be established:

Proposition 10. *Let* $a \in \mathbb{K}$ *and set* $\mathcal{H} := \{M \in M_3(\mathbb{K}) : \operatorname{tr} M = a\}$. *Then* $GL_3(\mathbb{K}) \subset \mathcal{H}^{(\infty)}$.

Proof. Notice that $\mathcal{H}$ is closed under conjugation hence $\mathcal{H}^{(\infty)}$ also is.

- Assume first that $\#\mathbb{K} > 2$. Then the union of the conjugacy classes of $\mathrm{Diag}(\lambda, 1, 1)$ for $\lambda \in \mathbb{K} \smallsetminus \{0, 1\}$ generates[1] the group $\mathrm{GL}_3(\mathbb{K})$. Notice that this subset is closed under inversion hence every matrix of $\mathrm{GL}_3(\mathbb{K})$ is a product of matrices in this subset.
  For every $\lambda \in \mathbb{K} \smallsetminus \{0, 1\}$, remark that
  $$\begin{bmatrix} a-1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & \lambda & 0 \\ 1 & a-1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & (\lambda+1)(a-1) & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \mathrm{Diag}(\lambda, 1, 1),$$
  hence $\mathrm{Diag}(\lambda, 1, 1)$ belongs to $\mathcal{H}^{(\infty)}$. This shows $\mathrm{GL}_3(\mathbb{K}) \subset \mathcal{H}^{(\infty)}$.
- Assume now $\#\mathbb{K} = 2$. Then every matrix of $\mathrm{GL}_3(\mathbb{K}) = \mathrm{SL}_3(\mathbb{K})$ is a product of matrices all similar to the transvection matrix $T := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (see [3] Proposition 9.1 p.541). If $a = 1$, we then see that
  $$T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathcal{H}^{(2)}.$$

[1] By [3] Proposition 9.1 p.541, it suffices to prove that some transvection matrix is a product of matrices of the aforementioned set. Choosing $\lambda \in \mathbb{K} \smallsetminus \{0, 1\}$, we see that $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 1-\lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \lambda^{-1} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with $\begin{bmatrix} \lambda^{-1} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \mathrm{Diag}(\lambda^{-1}, 1, 1)$ and $\begin{bmatrix} \lambda & 1-\lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \mathrm{Diag}(\lambda, 1, 1)$.

If $a=0$, we write:

$$T=\begin{bmatrix}0 & 1 & 1\\ 0 & 0 & 1\\ 1 & 0 & 0\end{bmatrix}\times\begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\in\mathcal{H}^{(2)}.$$

In any case, we deduce that $\mathrm{GL}_3(\mathbb{K})\subset\mathcal{H}^{(\infty)}$.

□

This completes the proof of Theorem 6 by induction.

4 Products of two matrices from an hyperplane

In this section, we consider a (linear) hyperplane H of $\mathrm{M}_n(\mathbb{K})$. If $n\geq 3$, then Theorem 2 shows that every matrix of $\mathrm{M}_n(\mathbb{K})$ is a product of matrices from H (possibly with a large number of factors). Here, we will see that actually two matrices always suffice in the product. As a warm up, we start by considering the case $n=2$ and by classifying all the counter-examples.
The following basic lemma of affine geometry will be of constant use:

Lemma 11. *Let F be a linear hyperplane of a vector space E, and $\mathcal{G}$ be an affine subspace of E with translation vector space G. If $F\cap\mathcal{G}=\emptyset$, then $G\subset F$.*

Proof. Assume $G\not\subset F$. Then $F+G=E$ since F is a linear hyperplane of E. Choosing $a\in\mathcal{G}$ and writing it $a=x+y$ for some $(x,y)\in F\times G$, we then see that $a-y\in F\cap\mathcal{G}$, hence $F\cap\mathcal{G}\neq\emptyset$. □

4.1 The case $n=2$

Here, we prove the following result:

Proposition 12. *Let H be a linear hyperplane of $M_2(\mathbb{K})$. Then every matrix of $M_2(\mathbb{K})$ is a product of two elements of H unless H is conjugate to one of the following hyperplanes*

$$H_0:=\left\{\begin{bmatrix}0 & b\\ a & c\end{bmatrix}\ \mid\ (a,b,c)\in\mathbb{K}^3\right\}\quad\textit{and}\quad T_2^+(\mathbb{K}):=\left\{\begin{bmatrix}a & b\\ 0 & c\end{bmatrix}\ \mid\ (a,b,c)\in\mathbb{K}^3\right\}.$$

Remark 2. Since $T_2^+(\mathbb{K})$ is a strict subalgebra of $\mathrm{M}_2(\mathbb{K})$, it clearly does not verify the result under scrutiny, and neither does any of its conjugate hyperplanes.
On the other hand, the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ cannot be decomposed as $A = BC$ for some pair $(B, C) \in H_0^2$. If indeed it could, then C would be non-singular, hence $C^{-1} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ for some triple $(a, b, c) \in \mathbb{K}^3$ with $b \neq 0$ and $c \neq 0$, and equating B with AC^{-1} would yield a contradiction (this would mean B has $c \neq 0$ as entry at the $(1, 1)$ spot).

Proof of Proposition 12. We assume H is neither conjugate to H_0 nor to $T_2(\mathbb{K})^+$. Choose an non-zero matrix A in the line $H^\perp$. Then A is conjugate to neither $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ nor to $\begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}$ for some $\lambda \neq 0$. This shows A is non-singular (if not, then A has rank 1 hence is conjugate to one of the aforementioned matrices). We let $M \in \mathrm{M}_2(\mathbb{K}) \smallsetminus \{0\}$ and try to decompose M as a product of two matrices in H.

- *The case M is non-singular.*
  For $N \in \mathrm{M}_2(\mathbb{K})$, we let $\mathrm{Com}(N)$ denote its matrix of cofactors. The map $N \mapsto \mathrm{Com}(N)$ is a linear automorphism of $\mathrm{M}_2(\mathbb{K})$, hence
  $$V := \left\{ M\,\mathrm{Com}(N)^T \mid N \in H \right\}$$
  is a hyperplane of $\mathrm{M}_2(\mathbb{K})$. If $V \cap H$ contains a non-singular matrix B, then we have a matrix $C \in H$ such that $M\ \mathrm{Com}(C)^T = B$, hence C is non-singular and $M = B\left(\frac{1}{\det(C)} \cdot C\right)$ belongs to $H^{(2)}$.
  Assume now that all the matrices in $V \cap H$ are singular. Since $\dim(V \cap H) \geq 2$, we deduce that H contains a two-dimensional singular linear subspace (i.e. one that contains no non-singular matrix). Replacing H with a conjugate hyperplane, we may use Lemma 32.1 of [5] and assume, without loss of generality, that H contains one of the planes
  $$\left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \mid (a, b) \in \mathbb{K}^2 \right\} \quad \text{or} \quad \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid (a, b) \in \mathbb{K}^2 \right\}.$$
  However, in the first case, the first row of A is zero, and in the second case, the first column of A is zero, contradicting the non-singularity of A. This completes the case M is non-singular.

- *The case M is singular.*
Then $\operatorname{rk} M = 1$ and we may choose a non-zero vector $e_1 \in \operatorname{Ker} M$ and extend it into a basis (e_1, e_2) of $\mathbb{K}^2$. Since $\{N \in \mathrm{M}_2(\mathbb{K}) : \; e_1 \in \operatorname{Ker} N\}$ is a linear plane, it has a common non-zero matrix C with H.
We now search for some $B \in H$ satisfying $M = B\,C$.
First of all, since $\operatorname{rk} C = \operatorname{rk} M$ and $e_1 \in \operatorname{Ker} C$, there is some $B_0 \in \mathrm{M}_2(\mathbb{K})$ such that $M = B_0\,C$. Then $\mathcal{P} := \bigl\{B \in \mathrm{M}_2(\mathbb{K}) : \; B\,C = M\bigr\}$ is a plane with translation vector space $P := \bigl\{B \in \mathrm{M}_2(\mathbb{K}) : \; B\,C = 0\bigr\}$.
If $\mathcal{P} \cap H \neq \emptyset$, then we find some $B \in H$ such that $M = B\,C$. If not, Lemma 11 would show that $P \subset H$, which would yield the same contradiction as in the case M is non-singular (we would find that A is singular). This completes the case M is singular.

$\square$

4.2 The case $n \geq 3$

Here, we assume $n \geq 3$, we let H be a linear hyperplane of $\mathrm{M}_n(\mathbb{K})$, and we choose a non-zero matrix A in $H^\perp$. Letting $M \in \mathrm{M}_n(\mathbb{K}) \smallsetminus \{0\}$, we try to decompose M as the product of two matrices in H.

4.2.1 The case M is singular

Up to conjugation by a well-chosen non-singular matrix, we may assume the first row of A is non-zero. We denote by $(e_1, \dots, e_n)$ the canonical basis of $\mathbb{K}^n$. The basic idea is to find a matrix C in H with the same kernel as M, and then another $B \in H$ such that $A = B\,C$ (notice the similarity with the case $n = 2$). Set $p := \operatorname{rk} M$, so that $1 \leq p < n$.

- The set
$$V := \bigl\{C \in \mathrm{M}_n(\mathbb{K}) : \operatorname{Ker} M \subset \operatorname{Ker} C \text{ and } \operatorname{Im} C \subset \operatorname{span}(e_2, \dots, e_n)\bigr\}$$
is a linear subspace of $\mathrm{M}_n(\mathbb{K})$ with dimension $(n-1)\,p$, and $\forall C \in V, \; \operatorname{rk} C \leq p$.

- It follows that $V \cap H$ has a dimension greater than or equal to $(n-1)p - 1$ and $\forall C \in V \cap H, \; \operatorname{rk} C \leq p$. Notice that $V \cap H$ is naturally isomorphic to a linear subspace of $\mathrm{M}_{n-1,p}(\mathbb{K})$ (through a rank-preserving map). If $V \cap H$ contained no rank p matrix, the Flanders-Meshulam theorem [4]

would show that $\dim(V \cap H) \leq (n-1)(p-1)$. However, since $n > 2$, one has $(n-1)(p-1) < np - p - 1$, hence $V \cap H$ contains a rank p matrix C. Therefore, $\operatorname{rk} M = \operatorname{rk} C$ and $\operatorname{Ker} M \subset \operatorname{Ker} C$, thus $\operatorname{Ker} M = \operatorname{Ker} C$ and it follows that $M = B_0\, C$ for some $B_0 \in \mathrm{M}_n(\mathbb{K})$.

- Define then the affine subspace $\mathcal{P} := \bigl\{B \in \mathrm{M}_n(\mathbb{K}) : \; B\,C = M\bigr\}$ with translation vector space $P := \bigl\{B \in \mathrm{M}_n(\mathbb{K}) : \; B\,C = 0\bigr\}$. By a *reductio ad absurdum*, let us assume that $\mathcal{P} \cap H = \emptyset$. Then Lemma 11 shows that $P \subset H$. However, since $\operatorname{Im} C \subset \operatorname{span}(e_2, \dots, e_n)$, it would follow that for any $C_1 \in \mathrm{M}_{n,1}(\mathbb{K})$, the matrix $\begin{bmatrix} C_1 & 0 & \cdots & 0 \end{bmatrix}$ would belong to H. This would entail that the first row of A is zero, in contradiction with our first assumption. We conclude that $\mathcal{P} \cap H \neq \emptyset$, which provides some $B \in H$ such that $M = B\,C$.

This shows that $M \in \mathcal{H}^{(2)}$ whenever M is singular.

4.2.2 The case M is non-singular

We will actually prove a somewhat stronger statement:

Proposition 13. *Let H_1 and H_2 be two linear hyperplanes of $M_n(\mathbb{K})$, with $n \geq 3$. Then there is a non-singular matrix $P \in H_1$ such that $P^{-1} \in H_2$.*

Before proving this, we readily show how this solves our problem. Since M is non-singular, $M^{-1}\,H$ is a linear hyperplane of $\mathrm{M}_n(\mathbb{K})$. Applying Proposition 13 to the hyperplanes H and $M^{-1}H$ yields a non-singular matrix $P \in H$ such that $P^{-1} \in M^{-1}H$. Therefore $P^{-1} = M^{-1}C$ for some $C \in H$, which shows $M = C\,P \in H^{(2)}$.

Proof of Proposition 13. We will use a *reductio ad absurdum* by assuming that no non-singular matrix $P \in H_1$ satisfies $P^{-1} \in H_2$.
Choose A_1 and A_2 respectively in $H_1^{\perp} \smallsetminus \{0\}$ and $H_2^{\perp} \smallsetminus \{0\}$. We will use the block decompositions:

$$A_1 = \begin{bmatrix} \alpha & L_1 \\ C_1 & M_1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} \beta & L_2 \\ C_2 & M_2 \end{bmatrix}$$

where $(\alpha, \beta) \in \mathbb{K}^2$, $(L_1, L_2) \in \mathrm{M}_{1,n-1}(\mathbb{K})^2$, $(C_1, C_2) \in \mathrm{M}_{n-1,1}(\mathbb{K})^2$ and $(M_1, M_2) \in \mathrm{M}_{n-1}(\mathbb{K})^2$.

To start with :

$$\text{We assume } C_1 \neq 0.$$

We will then prove that $C_2 = 0$ and $M_2 = 0$.
Let $Q \in \mathrm{GL}_{n-1}(\mathbb{K})$. For $X \in \mathrm{M}_{1,n-1}(\mathbb{K})$, set

$$f(X) := \begin{bmatrix} 1 & X \\ 0 & Q \end{bmatrix} \in \mathrm{GL}_n(\mathbb{K}),$$

the inverse of which is

$$f(X)^{-1} = \begin{bmatrix} 1 & -XQ^{-1} \\ 0 & Q^{-1} \end{bmatrix}.$$

Since $C_1 \neq 0$ and $n \geq 3$, there exists $X_0 \in \mathrm{M}_{1,n-1}(\mathbb{K}) \setminus \{0\}$ such that $f(X_0) \in H_1$. Set then $F := \bigl\{X \in \mathrm{M}_{1,n-1}(\mathbb{K}) : \; XC_1 = 0\bigr\}$, so that $f(X_0 + X) \in H_1$ for every $X \in F$. Then $\mathcal{G} := \bigl\{f(X_0 + X)^{-1} \mid X \in F\bigr\}$ is an affine subspace of $\mathrm{M}_n(\mathbb{K})$ with translation vector space

$$\left\{\begin{bmatrix} 0 & -XQ^{-1} \\ 0 & 0 \end{bmatrix} \mid X \in F\right\}.$$

By our initial assumption, one must have $\mathcal{G} \cap H_2 = \emptyset$, hence Lemma 11 shows that the translation vector space of $\mathcal{G}$ is included in H_2, which proves

$$\forall X \in \mathrm{M}_{1,n-1}(\mathbb{K}), \; XC_1 = 0 \Rightarrow XQ^{-1}C_2 = 0.$$

Since this holds for every non-singular Q, since $\mathrm{GL}_{n-1}(\mathbb{K})$ acts transitively on $\mathrm{M}_{n-1,1}(\mathbb{K}) \setminus \{0\}$, and $F \neq \{0\}$ (because $C_1 \neq 0$ and $n \geq 3$), we deduce that

$$C_2 = 0.$$

We now assume $M_2 \neq 0$ and prove that it leads to a contradiction. The matrix Q may now be chosen such that $f(0)^{-1} \in H_2$. Indeed, by Dieudonné's theorem for affine subspaces [1], the hyperplane of $\mathrm{M}_{n-1}(\mathbb{K})$ defined by the equation $\operatorname{tr}(M_2 N) = -\beta$ contains a non-singular matrix, and it suffices to choose Q as its inverse. Since $C_2 = 0$, we now have $f(X_0)^{-1} \in H_2$ which is a contradiction because $f(X_0) \in H_1$. We have thus proven:

$$M_2 = 0.$$

Let us sum up:

$$\text{If } e_1 \text{ is not an eigenvector of } A_1, \text{ then } \operatorname{Im} A_2 \subset \operatorname{span}(e_1).$$

Since the assumptions are unaltered by simultaneously conjugating H_1 and H_2 by an arbitrary non-singular matrix, we deduce:

For every non-zero vector $x \in \mathbb{K}^n$ which is not an eigenvector of A_1, one has $\operatorname{Im} A_2 \subset \operatorname{span}(x)$.

However $A_2 \neq 0$. It follows that, given two linearly independent vectors of $\mathbb{K}^n$, one must be an eigenvector of A_1. Obviously, this shows that A_1 is diagonalisable. Assume now that A_1 is not a scalar multiple of I_n.

- If $\# \mathbb{K} \geq 3$, then we may choose eigenvectors x and y of A_1 associated to distinct eigenvalues, choose $\lambda \in \mathbb{K} \smallsetminus \{0,1\}$, and notice that the vectors $x+y$ and $x+\lambda.y$ are linearly independent although none is an eigenvector of A_1.
- Assume now $\# \mathbb{K} = 2$ and choose a linearly independent triple (x,y,z) and a pair $(\lambda,\mu) \in \mathbb{K}^2$ of distinct scalars such that x,y,z are eigenvectors of A_1 respectively associated to the eigenvalues λ,λ,μ: then $x+z$ and $y+z$ are linearly independent and none is an eigenvector of A_1.

We deduce that A_1 is a scalar multiple of I_n. Since the pair (A_2,A_1) satisfies the same assumptions as (A_1,A_2), we also find that A_2 is a scalar multiple of I_n, hence $H_1 = H_2 = \mathfrak{sl}_n(\mathbb{K})$. Finally, the permutation matrix $P := E_{1,n} + \sum_{j=1}^{n-1} E_{j+1,j}$ belongs to $\mathfrak{sl}_n(\mathbb{K})$, and so does its inverse P^T. This is the final contradiction, which proves our claim. $\square$

This completes our proof of Theorem 3.
The reader will check that the preceding arguments may be generalized effortlessly so as to yield:

Theorem 14. *Let $n \geq 3$ be an integer, and H_1 and H_2 be two linear hyperplanes of $M_n(\mathbb{K})$. Then every $A \in M_n(\mathbb{K})$ splits as $A = B\,C$ for some $(B,C) \in H_1 \times H_2$.*

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